

On a family of exact solutions for a nonlinear diffusion equation

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August 11, 2014

Abstract

We present a complete description of the similarity solutions $u_\alpha(x, t) = t^{-\alpha/2} f(\|x\|/\sqrt{t}; \alpha)$ for the following nonlinear diffusion equation

$$u_t + \gamma|u_t| = \Delta u \quad (-1 < \gamma < 1)$$

The behaviors of these solutions are obtained through the explicit representation of $f(\eta; \alpha)$, in terms of Kummer and Tricomi functions. Considering results about confluent hypergeometric functions, new methods to describe asymptotic and oscillatory behaviors of the similarity solutions are obtained. We prove that there exists an increasing and unbounded sequence of positive similarity exponents such that the associated profile f has a gaussian rate decay. These special similarity exponents are related with the zeros of Kummer and Tricomi functions. Finally, we indicate how to extend our results on more general nonlinear diffusion equations.

bfkeywords: Similarity solutions, Kummer functions, Tricomi functions, nonlinear diffusion equations

1 Introduction

In physics and engineering problems finding similarity solutions is common. This feature is a consequence of symmetries in the underlying models used to solve these problems. It is known that these special functions are frequently used to describe the intermediate asymptotics of general solutions, i.e., behaviors in regions where the dependence with initial and/or boundary conditions already disappears [6, 8].

For nonlinear PDEs, the similarity solutions are also used to study the development of singularities and other class of particular phenomena such as blow-up or extinction in finite time [11]. Thus, there is an extensive literature on the relation between similarity solutions and behaviors of general solutions and, therefore, the analysis and description of these special class of solutions have a practical relevant importance from different points of view.

In this paper, we study the following nonlinear diffusion equation

$$u_t + \gamma|u_t| = \Delta u \quad \text{in } \mathbb{R}^N \times]0, \infty[, \quad (1)$$

where $-1 < \gamma < 1$. This equation was formulated in [4] as a nonlinear filtration theory model and since its presentation it has been studied in different framework in applied mathematics [8, 17, 20] and considered as a simple model of nonlinear parabolic equation [3, 10, 14, 15].

Our main objective is to obtain, through explicit representations, asymptotic and oscillatory behaviors of the similarity solutions

$$u_\alpha(x, t) = t^{-\alpha/2} f(\eta; \alpha), \quad \text{with } \eta = \|x\|/t^{1/2} \text{ and } \alpha > 0. \quad (2)$$

The analysis is centered on the profiles $f(\eta; \alpha)$ which are studied as solutions of a specific nonlinear Cauchy problem obtained when (2) is substituted into (1). The Cauchy problem to $f(\eta; \alpha)$ is described by a nonlinear ODE defined by a piecewise linear discontinuous function. Thus, the behaviors of $f(\eta; \alpha)$ are studied considering a collection of linear Cauchy problems.

Following ideas from [5, 10, 16, 18], we obtain a closed representation for (2) in terms of confluent hypergeometric functions, and oscillatory and asymptotic behaviors are obtained using the asymptotic representations of Kummer and Tricomi functions.

Our first main result reads as follows.

Theorem 1.1 *There exists an increasing and unbounded sequence of similarity exponents*

$$0 < \alpha_0 < \alpha_1 < \alpha_2 < \dots \quad (3)$$

such that $f(\eta; \alpha_k) \sim \eta^{\alpha_k - N} e^{-\frac{(1+(-1)^k \gamma)}{4} \eta^2}$ for η large. When $\alpha \neq \alpha_k$, the asymptotic behavior is given by $f(\eta; \alpha) \sim \eta^{-\alpha}$.

We show that the exponents in (3) are related with zeros of confluent hypergeometric functions. Specifically, the exponent α_m in (3) are determined by the analysis of the following system

$$\begin{aligned} M\left(\frac{N-\alpha}{2} - 1, \frac{N}{2}; \frac{(1-\gamma)}{4} \eta_1^2\right) &= 0 \\ U\left(\frac{N-\alpha}{2} - 1, \frac{N}{2}; \frac{(1+(-1)^m \gamma)}{4} \eta_{m+1}^2\right) &= 0 \quad (4) \\ M\left(\frac{N-\alpha}{2} - 1, \frac{N}{2}; \frac{(1+(-1)^m \gamma)}{4} \eta_{m+1}^2\right) \cdot \Gamma(\{N - \alpha\}/2) \cdot (-1)^m &> 0 \end{aligned}$$

obtaining a complete generalization of the results presented in [5, 16] for the exponent α_0 for $N = 1$ and N arbitrary, respectively.

Assuming that there exists exactly $m + 1$ roots $\eta_1 < \eta_2 < \dots < \eta_m < \eta_{m+1}$ for $\eta f' + \alpha f = 0$, in (4), the values η_1 and η_{m+1} denotes the smallest and largest roots, respectively. We use $M(a, b; z)$, $U(a, b; z)$ to denote the Kummer and Tricomi functions and Γ is the symbol for the Gamma-Euler function.

To complete the analysis about the behaviors of $f(\eta; \alpha)$, in our second main result the following oscillatory behaviors are presented.

Theorem 1.2 *Let α_k be a similarity exponent in Theorem 1.1. For each α such that $\alpha_k < \alpha \leq \alpha_{k+1}$, the profiles $f(\eta; \alpha)$ have exactly k zeros in $[0, \infty[$. If $\alpha \leq \alpha_0$ follows $f(\eta; \alpha) > 0$.*

This paper is organized as follows: In Section 2 we present results about (1) and a sequence of lemmas necessary for our proof on the oscillatory behavior of $f(\eta; \alpha)$. The method for the representation of similarity solutions in terms of Kummer and Tricomi functions is presented in Section 3. Theorems 1.1 and 1.2 are proved in Section 4, and the relation between the spatial decays of f and the zeros of Kummer and Tricomi functions is showed. Finally, in Section 5, we present how Theorems 1.1 and 1.2 can be extended to more general nonlinear diffusion equations.

2 Preliminaries

2.1 About the Barenblatt equation

Equation (1), commonly called Barenblatt equation, was formulated in [4] as a simple model to study the filtration of a low compressibility fluid through porous elasto-plastic media where deformations are irreversible. Such behavior is modeled by considering a piezo-conductivity coefficient which can take different values. Therefore, according to Darcy's law the following is observed

$$p_t = \kappa(p_t) \Delta p, \quad \kappa(z) = \begin{cases} \kappa^+ & (z > 0) \\ \kappa^- & (z \leq 0) \end{cases} \quad (5)$$

where κ^- and κ^+ depends upon the physical parameters of the model such as: coefficient of permeability; compressibility of the porous medium; compressibility of the fluid and others.

Considering the following transformations:

$$\gamma = \frac{\kappa^- - \kappa^+}{\kappa^- + \kappa^+} \quad \text{and} \quad t = \frac{1}{2} \left(\frac{1}{\kappa^+} - \frac{1}{\kappa^-} \right) t^*,$$

equation (5) can be written as (1), where $u(x, t^*) = p(x, t)$ (omitting the asterisk in (1)), satisfying directly the condition $-1 < \gamma < 1$. For further details about the derivation and physical consideration of this equation see [4, 7, 8] and their references.

Recently, equation (5) was considered as a model in sublinear expectation theory where the notions of G-Normal distribution, G-expectation and G-Brownian motion are introduced through the nonlinear parabolic equation

$$u_t - \frac{1}{2}((u_{xx})^+ - \sigma^2(u_{xx})^-) = 0; \quad 0 \leq \sigma \leq 1 \quad (6)$$

with $a^+ = \max\{0, a\}$, $a^- = (-a)^+$, i.e. $\gamma < 0$ in (1). In the framework of sublinear expectation theory, equation (6) is often called G-Heat equation. More details about its application can be studied in [13, 20, 21].

Due their simple mathematical structure, equation (1) often has been considered as practical example of nonlinear diffusion equations where is possible to use different tools for the analysis. The mathematical analyzes on (1) are related principally with the similarity solution associated with $\alpha = \alpha_0$, so-called anomalous exponent [3, 12].

Recently, Eq. (1) was used to study qualitative results related with more general diffusion equations, specifically, for the study of large time geometrical properties [14] and for the study of a counterexample to regularity for fully nonlinear equations [10].

Finally, we remark that in [13] an explicit family of similarity solution of (6) with polynomial initial condition $u(x, 0) = x^n$ was presented. The author use a matching between linear problems, similar to our method to solve (1).

2.2 Nonlinear ODE for the profiles $f(\eta; \alpha)$

If we assume that the function in (2) is a classical solution of (1), when we consider $\eta > 0$ such that $(\alpha f + \eta f) > 0$, the following ODE is obtained:

$$f'' + \frac{N-1}{\eta} f' = -\frac{(1-\gamma)}{2}(\alpha f + \eta f),$$

while in cases $(\alpha f + \eta f) < 0$ follows

$$f'' + \frac{N-1}{\eta} f' = \frac{(1+\gamma)}{2}(\alpha f + \eta f).$$

We note that in the formulation of the ODE we need the following functions:

$$\sigma(z) = \begin{cases} -(1-\gamma)/2 & (z > 0) \\ -(1+\gamma)/2 & (z \leq 0) \end{cases} \quad (7)$$

On the other hand, for the regularity of $u_\alpha(x, t)$ at $x = 0$, it's necessary that $f'(0) = 0$. Notice that, when multiplying (1) by a positive constant $A > 0$, we

get the solution $Au_\alpha(x, t)$ and therefore, the value $f(0) = A > 0$ can be chosen arbitrarily. The profile $f(\eta)$ is conveniently normalized by the relation $f(0) = 1$. Thus, the profile $f(\eta; \alpha)$ satisfies the following nonlinear Cauchy problem

$$\begin{cases} f'' + \frac{N-1}{\eta} f' &= \sigma(\alpha f + \eta f')(\alpha f + \eta f') \\ f(0) &= 1 \\ f'(0) &= 0 \end{cases} \quad (8)$$

The existence and uniqueness follows from the fact that $\sigma(z)$ is a piecewise linear function. Thus, for each $\alpha > 0$ there exist an unique solution for (8) and using these profiles we obtain a one-parameter family of similarity solutions (2) for the equation (1).

From the form of the ODE in (8), the solution can be obtained using a collection of linear Cauchy problems, each one formulated in different ranges of η . Each of these ranges correspond to an interval where the term $\eta f'(\eta; \alpha) + \alpha f(\eta; \alpha)$ does not change sign. Hence, considering the n positive roots $\eta_1 < \eta_2 < \dots < \eta_n$ of

$$\eta f'(\eta; \alpha) + \alpha f(\eta; \alpha) = 0, \quad (9)$$

the ranges of linear systems are: $]0, \eta_0[,]\eta_0, \eta_1[, \dots,]\eta_{n-1}, \eta_n[,]\eta_n, \infty[$ (see (18) below).

Through an appropriate transformation, the ODE in each linear problem is associated with the following Kummer ODE:

$$zu'' + ((N/2) - z)u' - (\alpha/2)u = 0. \quad (10)$$

Hence, the profile f can be represented through of the solutions of (10). Eq. (10) has an entire solution called Kummer function, defined by

$$M(\alpha/2, N/2; z) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha/2)_k}{(N/2)_k} \frac{z^k}{k!}, \quad (11)$$

where $(\lambda)_k$ is the Pochhammer symbol:

$$(\lambda)_0 = 1, \quad (\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \lambda(\lambda + 1)(\lambda + 2) \dots (\lambda + k - 1), \quad k = 1, 2, \dots, \quad (12)$$

The Kummer function is also called confluent hypergeometric function (of the first kind) and it's also denoted by ${}_1F_1(a, b; z)$.

We remark that when $\alpha < N - 2$ follows $\alpha f + \eta f' > 0$ and therefore f is obtained solving (2.2). Thus, if $\alpha < N - 2$, we have the following representation

$$f(\eta; \alpha) = M\left(\frac{\alpha}{2}, \frac{N}{2}, -\frac{(1-\gamma)}{4}\eta^2\right), \quad (13)$$

satisfying $f(\eta; \alpha) > 0$. This result is obtained considering $z = -(1-\gamma)\eta^2/4$ and the transformation $g(z) = f(\eta; \alpha)$. We remark that the existence of roots for $\eta f' + \alpha f = 0$ for each $\alpha > N - 2$ is studied in Lemma 2.1 (below).

On the other hand, in the linear case $\gamma = 0$, the family of self-similar solutions with $\alpha > 0$ are described through the Kummer functions (11). Specifically, if $\gamma = 0$ the profile $f(\eta; \alpha)$ is described as follows:

$$f_{heat}(\eta; \alpha) = M\left(\frac{\alpha}{2}, \frac{N}{2}; -\frac{\eta^2}{4}\right).$$

In cases $b - a = -l$ ($l = 0, 1, 2, \dots$), the Kummer function $M(a, b; z)$ has an exponential-type behavior to $z \rightarrow -\infty$. This result is directly verified using the Kummer transformation

$$\begin{aligned} M(a, b; z) &= e^{-z} M(b - a, b; -z) \\ &= e^{-z} \left\{ 1 + \sum_{k=0}^{\infty} \frac{(b - a)_k}{(b)_k} \frac{(-z)^k}{k!} \right\}. \end{aligned} \quad (14)$$

Now, from the definition of $(a)_k$ in (12), when $b - a = -l$ (with $l = 0, 1, 2, \dots$), we have $(b - a)_k = 0$ for each $k \geq l$ and therefore we obtain de exponential behavior. Using the representation (14), we note that $f_{heat}(\eta; \alpha)$ have an exponential-type decay when $\frac{\alpha - N}{2} = l$ ($l = 0, 1, 2, \dots$). Hence, when $\gamma = 0$ the similarity exponents α_l in (3) are represented by $\alpha_l = N + 2l$.

In the case where $\gamma \neq 0$, we need to use recurrence results and asymptotic representations of the Tricomi function or confluent hypergeometric function of the second kind, denoted by $U(\alpha/2, N/2; z)$, see for instance [1, 19].

2.3 Outline of the construction of the Similarity Solutions

We continue using $f(\eta; \alpha)$ to denote the solution of (8). If (9) has no solutions the representation for f is given in (13). In other case, we need solve more linear Cauchy problems.

Let $\eta_1 > 0$ be the first positive root of (9). Using the initial conditions for f , we have $\alpha f + \eta f' > 0$ when $\eta = 0$. Therefore for each η such that $0 < \eta < \eta_1$ follows $\alpha f + \eta f' > 0$. Let $f_1(\eta; \alpha)$ be the solution of the following linear problem:

$$\begin{cases} f_1'' + \frac{N-1}{\eta} f_1' &= -\frac{(1-\gamma)}{2} (\alpha f_1 + \eta f_1') \\ f_1(0) &= 1 \\ f_1'(0) &= 0 \end{cases} \quad (15)$$

As $\alpha f + \eta f' > 0$ when $0 < \eta < \eta_1$, from $\sigma(z)$ in (7), we have

$$f'' + \frac{N-1}{\eta} f' = -\frac{(1-\gamma)}{2} (\alpha f + \eta f') \quad (0 < \eta < \eta_1)$$

and thus, through classical uniqueness results on ODE, follows

$$f(\eta; \alpha) = f_1(\eta; \alpha), \quad \text{para } 0 \leq \eta \leq \eta_1. \quad (16)$$

Now, we continue in a similar way. Assume that (9) has exactly n positive roots, denoted by $\eta_1 < \eta_2 < \dots < \eta_n$. Through these roots, we describe different ranges where auxiliary linear Cauchy problems are defined. Let $f_2(\eta; \alpha)$ be the solution of the following Cauchy problem

$$\begin{cases} f_2'' + \frac{N-1}{\eta} f_2' &= -\frac{(1+\gamma)}{2}(\alpha f_2 + \eta f_2') & \eta > \eta_1 \\ f_2(\eta_1) &= f_1(\eta_1) \\ f_2'(\eta_1) &= f_1'(\eta_1) \end{cases} \quad (17)$$

From the definition of η_i , follows $\alpha f + \eta f' < 0$ in $\eta_1 \leq \eta \leq \eta_2$. Hence,

$$f_2'' + \frac{N-1}{\eta} f_2' = -\frac{(1+\gamma)}{2}(\alpha f + \eta f') \quad (\eta_1 < \eta < \eta_2)$$

with $f_2(\eta_1) = f(\eta_1)$ and $f_2'(\eta_1) = f'(\eta_1)$. Using uniqueness results, we have

$$f(\eta; \alpha) = f_2(\eta; \alpha), \quad \eta_1 \leq \eta \leq \eta_2.$$

Following the previous ideas, the profile $f(\eta; \alpha)$ is defined as piecewise function

$$f(\eta; \alpha) = \begin{cases} f_1(\eta; \alpha) & 0 \leq \eta \leq \eta_1 \\ f_2(\eta; \alpha) & \eta_1 \leq \eta \leq \eta_2 \\ \vdots & \\ f_n(\eta; \alpha) & \eta_{n-1} \leq \eta \leq \eta_n \\ f_{n+1}(\eta; \alpha) & \eta_n \leq \eta \end{cases} \quad (18)$$

where $f_m(\eta; \alpha)$ is the solution of the linear Cauchy problem:

$$\begin{cases} f_m'' + \frac{N-1}{\eta} f_m' &= -\frac{(1+(-1)^m \gamma)}{2}(\alpha f_m + \eta f_m') & \eta_{m-1} < \eta < \eta_m \\ f_m(\eta_{m-1}) &= f_{m-1}(\eta_{m-1}) \\ f_m'(\eta_{m-1}) &= f_{m-1}'(\eta_{m-1}) \end{cases}$$

The case $m = 1$ is given in (15) and in the case $m = n+1$ we consider $\eta_{n+1} = \infty$ in (2.3).

Finally, for each $m = 1, 2, \dots, n+1$, we consider the following change of variable and transformation:

$$z_m = -\frac{(1+(-1)^m \gamma)}{4} \eta^2; \quad g_m(z_m; \alpha) = f_m(\eta; \alpha) \quad (19)$$

Considering these substitutions, the ODE in (2.3) is rewritten as follows:

$$z_m g_m'' + ((N/2) - z_m) g_m' - (\alpha/2) g_m = 0, \quad (20)$$

i.e, a Kummer-type ODE with parameters $a = \alpha/2$ and $b = N/2$. Thus, the functions $f_m(\eta; \alpha)$ can be represented by the Kummer and Tricomi functions. For $m = 1$ we use (13) to represent $f_1(\eta; \alpha)$. Hence, knowing (16), from (13) we obtain the representation of $f(\eta; \alpha)$ when $\eta \in [0, \eta_1]$. In Section 3 we return with

the representation of similarity solutions through the confluent hypergeometric functions.

In regard to the matching technique between linear problems, in [10, 18] this technique was used to study similarity solutions of nonlinear problems related to (1). In [18] an equation with similar symmetries as (1) was studied (see (50) below). The author uses the behaviors of confluent hypergeometric functions to obtain the existence of an exponent such that the similarity solution vanishes at least exponentially fast at infinity.

On the other hand, in [10] a method to describe similarity solutions of a nonlinear diffusion problem was presented to obtain a counterexample for the regularity in nonlinear diffusion problems. In this article the authors showed a similar results as presented in Theorems 1.1 and 1.2 in our paper, but the results on similarity solutions are not obtained through the explicit representations.

To finish this section we present the following lemma where a sufficient condition for the existence of $\eta_1 < \infty$ is presented.

Lemma 2.1 *The equation $\alpha f + \eta f' = 0$ has solutions if and only if $\alpha > N - 2$.*

Proof: Using the representation (16) we know that the first positive solution of (9) is related with the equation

$$\eta f_1'(\eta; \alpha) + \alpha f_1(\eta; \alpha) = 0$$

where $f_1(\eta; \alpha)$ is solution of the linear problem (15). Considering $m = 1$ in (19), we get (10) and therefore the solution is represented as follows:

$$f_1(\eta; \alpha) = M\left(\frac{\alpha}{2}, \frac{N}{2}; -\frac{(1-\gamma)}{4}\eta^2\right),$$

with $M(a, b; z)$ defined in (11). Thus, using this representation, we notice that the root η_1 is determined by the following equation

$$\alpha M\left(\frac{\alpha}{2}, \frac{N}{2}; z\right) + 2zM'\left(\frac{\alpha}{2}, \frac{N}{2}; z\right) = 0, \quad (21)$$

with $z = -(1-\gamma)\eta^2/4$. Through the recurrence relation (see 13.4.10 in [1])

$$zM'(a, b; z) + aM(a, b; z) = aM(a+1, b; z), \quad (22)$$

equation (21) is rewritten as follows:

$$M\left(\frac{\alpha}{2} + 1, \frac{N}{2}; -\frac{(1-\gamma)}{4}\eta^2\right) = 0.$$

Considering Kummer's transformation

$$M(a, b; z) = e^z M(b-a, b; -z) \quad (23)$$

we obtain

$$M\left(\frac{N}{2} - \frac{\alpha}{2} - 1, \frac{N}{2}; \frac{(1-\gamma)}{4}\eta^2\right) = 0. \quad (24)$$

Thus, the existence of the root η_1 and its behaviours are related directly with the first positive zero of Kummer function $M(a, b; z)$ with $a = \frac{N}{2} - \frac{\alpha}{2} - 1$. From (11), we note that in cases $\frac{N}{2} - \frac{\alpha}{2} - 1 \leq 0$ the equation (24) has no positive solution, obtaining a sufficient condition for the non-existence of η_1 . Now, knowing that if $a < 0$ the function $M(a, N/2; z)$ has exactly $-\lfloor a \rfloor$ positive zeros, when $\alpha > N - 2$ the first root η_1 exists. \square

We note that the representation to f given in (13) is obtained when $\alpha \leq N - 2$. To respect the inequality $\alpha > N - 2$, in [15] (Theorem 2.3) is used to remark that $\alpha_0 > N - 2$. This result is direct from Lemma 2.1.

2.4 Auxiliary results on sign change of the profile

In this part we present some comments and auxiliary results to understand the reasons of the oscillatory behaviors of $f(\eta; \alpha)$. Moreover, the auxiliary results in this section are fundamental in our proofs to understand the representation of $f(\eta; \alpha)$ through the confluent hypergeometric functions and the relation between the asymptotic representation and the similarity exponents α_k in (3).

We continue assuming that (9) has exactly n positive solutions which are denoted by

$$\eta_1 < \eta_2 < \dots < \eta_n$$

Let us also assume that the roots η_m can be considered as regular functions of α . The dependence $\eta_m = \eta_m(\alpha)$ is verified directly through the representations of $f_m(\eta; \alpha)$ using confluent hypergeometric functions, see (38) below. On the other hand, from the representations given later in (33) and (38), the function

$$h(\eta, \alpha) = f(\eta; \alpha) \quad (25)$$

is continuous for each $(\eta, \alpha) \in \mathbb{R}^+ \times \mathbb{R}^+$. This result is used to prove the following lemma.

Lemma 2.2 *The functions $h_m(\alpha) = f(\eta_m; \alpha)$ do not change signs.*

Proof: We begin analyzing the sign of $h_1(\alpha)$. The proof is similar for the general case.

We know that $f_1(\eta_1; \alpha_*) > 0$ for some α_* . If we assume $f_1(\eta_1; \alpha^*) < 0$ for some $\alpha^* > \alpha_*$, through the continuity of function $h(\eta, \alpha)$ in (25) there must exist some $\tilde{\alpha}$ in $]\alpha_*, \alpha^*[$ such that $f_1(\eta_1; \tilde{\alpha}) = 0$. Directly from the definition of

$\eta_1 = \eta_1(\tilde{\alpha})$ we get $f'_1(\eta_1; \tilde{\alpha}) = 0$. Considering

$$\begin{cases} f'' + \frac{N-1}{\eta} f' &= -\frac{(1-\gamma)}{2}(\tilde{\alpha}f + \eta f) \\ f(\eta_1) &= f_1(\eta_1; \tilde{\alpha}) = 0 \\ f'(\eta_1) &= f'_1(\eta_1; \tilde{\alpha}) = 0 \end{cases}$$

and using existence and uniqueness results we get $f_1(\eta; \tilde{\alpha}) = 0$, obtaining a contradiction. Thus, the function $h_1(\alpha)$ does not change sign. In the general case $m = 1, 2, \dots, n$, the argument follows in a similar way and therefore $h_m(\alpha) = f_m(\eta_i; \alpha)$ do not change sign. \square

Knowing that f is defined as piecewise function, the next step is to study the oscillatory behavior in each range $] \eta_{m-1}, \eta_m[$.

Lemma 2.3 *The function $f(\eta; \alpha)$ changes sign exactly once in $] \eta_{m-1}, \eta_m[$.*

Proof: The result is obtained through contradiction method, studying the behavior of the following function

$$F(\eta) = \eta f'(\eta; \alpha) + \alpha f(\eta; \alpha). \quad (26)$$

Our analysis begins at interval $] \eta_1, \eta_2[$ where $F(\eta) < 0$. If we assume that $f(\eta; \alpha)$ changes sign more than once in $] \eta_1, \eta_2[$, then there exists at least one $\eta^* \in] \eta_1, \eta_2[$ such that $f(\eta^*; \alpha) = 0$ and $f'(\eta^*; \alpha) > 0$. Using (26) we get $F(\eta^*) > 0$, obtaining a contradiction. Now, we assume that $f(\eta; \alpha)$ does not change sign in $] \eta_1, \eta_2[$. Knowing that $f(\eta_1; \alpha) > 0$ we continue working under the assumption $f(\eta; \alpha) > 0$ in $] \eta_1, \eta_2[$ and therefore $f'(\eta; \alpha) < 0$ in $] \eta_1, \eta_2[$ (from $F(\eta) < 0$ in $] \eta_1, \eta_2[$).

Since (26), we have directly

$$\begin{aligned} \frac{dF}{d\eta} &= \eta f'' + (\alpha + 1)f' \\ &= \eta \left(f'' + \frac{N-1}{\eta} f' \right) + (\alpha - (N-2))f'. \end{aligned}$$

On the other hand, knowing that f is a solution to (8), in $] \eta_1, \eta_2[$ we get $\sigma(\alpha f + \eta f') = -\frac{1+\gamma}{2}$ and therefore

$$\frac{dF}{d\eta} = -\eta \frac{1+\gamma}{2} F + (\alpha - (N-2))f'.$$

This equation is written as follows

$$\frac{d}{d\eta} \left(e^{\frac{(1+\gamma)}{4}\eta^2} F(\eta) \right) = (\alpha - (N-2))e^{\frac{(1+\gamma)}{4}\eta^2} f'.$$

Integrating the equation above between η_1 and η_2 , and using $F(\eta_1) = F(\eta_2) = 0$, we get

$$\int_{\eta_1}^{\eta_2} (\alpha - (N-2))e^{\frac{(1+\gamma)}{4}\eta^2} f' d\eta = 0$$

Under the assumption $f'(\eta; \alpha) < 0$ in $]\eta_1, \eta_2[$ and knowing $\alpha > N - 2$, a contradiction is obtained. Thus, functions $f(\eta; \alpha)$ change sign exactly once in the interval $]\eta_1, \eta_2[$.
Finally, as

$$\text{Sign}(\eta f'_m + \alpha f_m) = (-1)^{m+1} \quad \text{when } \eta_{m-1} < \eta < \eta_m, \quad m = 1, 2, \dots, n.$$

and following previous arguments, for each $m = 2, 3, \dots, n$ we obtain the result in $]\eta_{m-1}, \eta_m[$. \square

A direct consequence of the result above is $f(\eta_m; \alpha) \cdot f(\eta_{m+1}; \alpha) < 0$. As $f(\eta_1; \alpha) > 0$, we get

$$\text{Sign}(f(\eta_m; \alpha)) = (-1)^{m+1} \quad \text{for each } m = 1, 2, \dots, n \quad (27)$$

The result in the following lemma will be used to characterize the decay rate of $f(\eta; \alpha)$ when $\eta \rightarrow \infty$

Lemma 2.4 *In $]\eta_n, \infty[$ the function $f(\eta; \alpha)$ does not change sign.*

Proof: The argument is similar as in the proof of Lemma 2.3. We continue using $F(\eta)$ given in (26) and using (27) we know that the sign of $F(\eta)$ is determinate by the parity of n . We begin considering $\alpha f + \eta f' < 0$ in $]\eta_n, \infty[$, i.e. n odd. The proof for the even case is similar.

From (27) we get $f(\eta_n) > 0$. If we assume that $f(\eta; \alpha)$ change sign more than once in $]\eta_n, \infty[$, then, there exists $\eta^* > \eta_n$ such that $f(\eta^*; \alpha) = 0$ and $f'(\eta^*; \alpha) > 0$. From the assumption $F(\eta) < 0$ a contradiction is obtained. Now, we assume that $f(\eta; \alpha)$ changes sign exactly once in $]\eta_n, \infty[$.

Let η_* be the point where $f(\eta_*; \alpha) = 0$. From the proof of Lemma 2.3, f can't change sign more than once, therefore $f(\eta; \alpha) < 0$ in $]\eta_*, \infty[$. As $f(\eta; \alpha) \rightarrow 0$ when $\eta \rightarrow \infty$, there exists a certain $\eta^* > \eta_*$ such that $f'(\eta^*; \alpha) = 0$ and $f'(\eta; \alpha) > 0$ when $\eta > \eta^*$.

Knowing that $f''(\eta; \alpha) \rightarrow 0$ when $\eta \rightarrow \infty$, using the ODE in (8) for $\eta > \eta^*$ and the definition in (26), we get:

$$\lim_{\eta \rightarrow \infty} F(\eta) = 0. \quad (28)$$

Similar to the proof for Lemma 2.3, considering the assumption $f'(\eta; \alpha) > 0$ in $]\eta_n, \infty[$ and knowing that $\alpha > N - 2$, we have $(\alpha + 1)f' > (N - 1)f'$. Thus:

$$\begin{aligned} \frac{dF}{d\eta} &> \eta \left(f'' + \frac{N-1}{\eta} f' \right) \\ &> -\eta \frac{(1+\gamma)}{2} (\alpha f + \eta f). \end{aligned}$$

From the assumption $F < 0$ we get:

$$-\frac{dF}{F} > \frac{(1+\gamma)}{2} \eta d\eta.$$

Since $F(\eta^*) = \alpha f(\eta^*; \alpha) + \eta^* f'(\eta^*; \alpha) = \alpha f(\eta^*; \alpha) < 0$, integrating in $[\eta^*, \eta[$ the inequality above, we obtain:

$$-F(\eta) > (-\alpha f(\eta^*; \alpha))e^{-\frac{(1+\gamma)}{4}(\eta^*)^2} e^{\frac{(1+\gamma)}{4}\eta^2}.$$

Considering $\eta \rightarrow \infty$ we obtain a contradiction with the fact (28). Thus, $f(\eta; \alpha)$ can not change sign in $[\eta_n, \infty[$. Finally, if we assume $\alpha f + \eta f' > 0$ in $]\eta_n, \infty[$, taking $\tilde{F}(\eta) = -F(\eta)$ we obtain similar contradictions. \square

3 Representation of the similarity solutions

In this part we detail the representations of f using the Kummer and Tricomi functions.

For simplicity, the method to describe the profiles $f(\eta; \alpha)$ is developed by separate according to the following cases: positive and sign change similarity solutions.

3.1 Positive similarity solutions

We continue working under the assumption $\alpha > N - 2$, therefore η_1 exists (see Lemma 2.1), in other case the representation of f is given by (13).

We begin considering the second range given in (17). Taking $m = 2$ in (19), $f_2(\eta; \alpha)$ can be represented by two linearly independent solutions of Kummer equations.

Knowing that (11) is the first solution, in this part we consider the Tricomi function $U(\alpha/2, N/2; z)$ as the second linearly independent solution of (10).

As (see formulae 13.1.22 [1])

$$\mathcal{W}\{M(\alpha/2, N/2; z), U(\alpha/2, N/2; z)\} = -\frac{\Gamma(N/2)}{\Gamma(\alpha/2)} z^{-N/2} e^z, \quad (29)$$

taking $\varphi_1(\eta) = M(\alpha/2, N/2; -\frac{(1+\gamma)}{4}\eta^2)$ and $\varphi_2(\eta) = U(\alpha/2, N/2; -\frac{(1+\gamma)}{4}\eta^2)$, complex solutions for the ODE in (17), the solution $f_2(\eta; \alpha)$ can be written as follows:

$$f_2(\eta; \alpha) = A_2(\alpha)M\left(\frac{\alpha}{2}, \frac{N}{2}; -\frac{(1+\gamma)}{4}\eta^2\right) + B_2(\alpha)U\left(\frac{\alpha}{2}, \frac{N}{2}; -\frac{(1+\gamma)}{4}\eta^2\right). \quad (30)$$

Here, we consider the principal branch for $U(\alpha/2, N/2; z)$, i.e. $\text{ph}z \in [0, \pi]$. Knowing that $f_2(\eta; \alpha)$ must satisfy $f_2(\eta_1; \alpha) = f_1(\eta_1; \alpha)$ and $f_2'(\eta_1; \alpha) = f_1'(\eta_1; \alpha)$, to determinate $A_2(\alpha)$ and $B_2(\alpha)$ we consider

$$\begin{pmatrix} M(\frac{\alpha}{2}, \frac{N}{2}; s_1) & U(\frac{\alpha}{2}, \frac{N}{2}; s_1) \\ M'(\frac{\alpha}{2}, \frac{N}{2}; s_1) & U'(\frac{\alpha}{2}, \frac{N}{2}; s_1) \end{pmatrix} \begin{pmatrix} A_2(\alpha) \\ B_2(\alpha) \end{pmatrix} = \begin{pmatrix} f_1(\eta_1; \alpha) \\ -\frac{f_1(\eta_1; \alpha)}{(1+\gamma)\eta_1} f_1'(\eta_1; \alpha) \end{pmatrix},$$

with $s_1 = -\frac{(1+\gamma)}{4}\eta^2$. On the other hands, as $\alpha f_1(\eta_1; \alpha) + \eta_1 f_1'(\eta_1; \alpha) = 0$, hence

$$A_2(\alpha) = \frac{f_1(\eta_1; \alpha)}{s_1 \mathcal{W}_1} (s_1 U'(\frac{\alpha}{2}, \frac{N}{2}; s_1) + \frac{\alpha}{2} U(\frac{\alpha}{2}, \frac{N}{2}; s_1))$$

and

$$B_2(\alpha) = \frac{f_1(\eta_1; \alpha)}{s_1 \mathcal{W}_1} (s_1 M'(\frac{\alpha}{2}, \frac{N}{2}; s_1) + \frac{\alpha}{2} M(\frac{\alpha}{2}, \frac{N}{2}; s_1)),$$

where \mathcal{W}_1 denote the Wronskian (29) evaluated at $\eta = \eta_1$.

The next step is to characterize the constants $A_2(\alpha)$ and $B_2(\alpha)$. We begin with $A_2(\alpha)$, where are used recurrence results of $U(a, b; z)$.

From the relation (see [1] 13.4.25)

$$aU(a, b; z) + zU'(a, b; z) = a(1 + a - b)U(a + 1, b; z),$$

follows

$$A_2(\alpha) = \frac{f_1(\eta_1; \alpha)}{s_1 \mathcal{W}_1} \frac{\alpha}{2} \left\{ 1 + \frac{\alpha - N}{2} \right\} U(\frac{\alpha}{2} + 1, \frac{N}{2}; s_1). \quad (31)$$

Now, we use recurrence relations of the Kummer function to characterize $B_2(\alpha)$.

From (22), we get

$$B_2(\alpha) = -\frac{f_1(\eta_1; \alpha)}{s_1 \mathcal{W}_1} \frac{\alpha}{2} M(\frac{\alpha}{2} + 1, \frac{N}{2}; s_1). \quad (32)$$

Therefore, considering $z = -\frac{(1+\gamma)}{4}\eta^2$, while $f > 0$, its representation is given by

$$f(\eta; \alpha) = \begin{cases} M(\frac{\alpha}{2}, \frac{N}{2}; z) & 0 \leq \eta \leq \eta_1 \\ A_2(\alpha)M(\frac{\alpha}{2}, \frac{N}{2}; \frac{1+\gamma}{1-\gamma}z) + B_2(\alpha)U(\frac{\alpha}{2}, \frac{N}{2}; \frac{1+\gamma}{1-\gamma}z) & \eta_1 \leq \eta \end{cases} \quad (33)$$

where $A_2(\alpha)$ and $B_2(\alpha)$ are defined in (31) and (32) respectively.

We note that, if η_1 is the unique solution to (9), we have $f > 0$. In other case we consider the result presented in Lemma 2.3. This is proved directly by the asymptotic representations of $f(\eta; \alpha)$ shown in the next section.

3.2 Similarity solutions with sign changes

To illustrate our method, we present the case $n = 2$ in (18), thus, $\alpha f + \eta f' = 0$ has two roots denoted by $\eta_1 < \eta_2$. Following similar ideas, the remaining cases are presented below.

Let $f(\eta; \alpha)$ be the solution of problem (8) given in (18) with $n = 2$.

Using the representation of $f_2(\eta; \alpha)$ in (30), Eq. (9) is written as follows

$$\begin{aligned} & \alpha \{ A_2(\alpha)M(\frac{\alpha}{2}, \frac{N}{2}; s_2) + B_2(\alpha)U(\frac{\alpha}{2}, \frac{N}{2}; s_2) \} - \\ & - s_2 \{ A_2(\alpha)M'(\frac{\alpha}{2}, \frac{N}{2}; s_2) + B_2(\alpha)U'(\frac{\alpha}{2}, \frac{N}{2}; s_2) \} = 0, \end{aligned} \quad (34)$$

with $s_2 = -\frac{(1+\gamma)}{4}(\eta_2)^2$. Following similar ideas as those shown in the descriptions of the constants $A_1(\alpha)$ and $B_1(\alpha)$, we prove that equation (34) can be written

$$A_2(\alpha)M\left(\frac{\alpha}{2} + 1, \frac{N}{2}; s_2\right) = -B_2(\alpha) \left\{1 + \frac{\alpha - N}{2}\right\} U\left(\frac{\alpha}{2} + 1, \frac{N}{2}; s_2\right). \quad (35)$$

Considering η_2 , the second root of $\eta f'_2(\eta; \alpha) + \frac{\alpha}{2}f_2(\eta; \alpha) = 0$, we get $s_2 = -\frac{(1-\gamma)}{4}(\eta_2)^2$ the root in (35). Notice that s_2 is described using the first root η_1 and α . From (24) and using the implicit function theorem, η_1 can be defined as regular function $\eta_1 = \eta_1(\alpha)$ and therefore $\eta_2 = \eta_2(\alpha)$.

We know that $f(\eta; \alpha) = f_3(\eta; \alpha)$ in $[\eta_2, \infty[$. Using (19) follows

$$f_3(\eta; \alpha) = A_3(\alpha)M\left(\frac{\alpha}{2}, \frac{N}{2}; -\frac{(1-\gamma)}{4}\eta^2\right) + B_3(\alpha)U\left(\frac{\alpha}{2}, \frac{N}{2}; -\frac{(1-\gamma)}{4}\eta^2\right) \quad (36)$$

Hence, following similar arguments as in the analysis for f_2 , we get:

$$\begin{aligned} A_3(\alpha) &= \frac{f_2(\eta_2; \alpha)}{s_2 \mathcal{W}_2} \frac{\alpha}{2} \left\{1 + \frac{\alpha - N}{2}\right\} U\left(\frac{\alpha}{2} + 1, \frac{N}{2}; s_2\right) \\ B_3(\alpha) &= -\frac{f_2(\eta_2; \alpha)}{s_2 \mathcal{W}_2} \frac{\alpha}{2} M\left(\frac{\alpha}{2} + 1, \frac{N}{2}; s_2\right), \end{aligned} \quad (37)$$

where $\varphi_1(\eta) = M(\alpha/2, N/2; -\frac{(1-\gamma)}{4}\eta^2)$, $\varphi_2(\eta) = U(\alpha/2, N/2; -\frac{(1-\gamma)}{4}\eta^2)$, and \mathcal{W}_2 is the Wronskian between φ_1 and φ_2 at $\eta = \eta_2$.

On the other hand, since $A_3(\alpha)$ depends of η_2 , then $A_3(\alpha)$ depends of η_1 (similar to $B_3(\alpha)$). For the general case, consider the n positive roots: $\eta_1 < \eta_2 < \dots < \eta_n$ of equation (9). For each $m = 1, 2, \dots, n+1$, consider $f_m(\eta; \alpha)$ as (18). $f_1(\eta; \alpha)$ is defined by (13) and in the cases $m \geq 2$, we have

$$f_m(\eta; \alpha) = A_m(\alpha)M\left(\frac{\alpha}{2}, \frac{N}{2}; z_m\right) + B_m(\alpha)U\left(\frac{\alpha}{2}, \frac{N}{2}; z_m\right), \quad (38)$$

where $z_m = -\frac{(1+(-1)^m\gamma)}{4}\eta^2$ with $\eta_{m-1} \leq \eta \leq \eta_m$. Likewise, we define $f_{n+1}(\eta; \alpha)$ for $\eta \in [\eta_n, \infty[$. In a similar way as in the statement of (37), for the general case, we have:

$$\begin{aligned} A_m(\alpha) &= \frac{f_{m-1}(\eta_{m-1}; \alpha)}{s_{m-1} \mathcal{W}_{m-1}} \frac{\alpha}{2} \left\{1 + \frac{\alpha - N}{2}\right\} U\left(\frac{\alpha}{2} + 1, \frac{N}{2}; s_{m-1}\right) \\ B_m(\alpha) &= -\frac{f_{m-1}(\eta_2; \alpha)}{s_{m-1} \mathcal{W}_{m-1}} \frac{\alpha}{2} M\left(\frac{\alpha}{2} + 1, \frac{N}{2}; s_{m-1}\right), \end{aligned} \quad (39)$$

where $s_{m-1} = -\frac{(1+(-1)^m\gamma)}{4}(\eta_{m-1})^2$ and \mathcal{W}_{m-1} the Wronskian (29) at η_{m-1} . Here, η_{m-1} is the $m-1$ root of $\eta f'_{m-1} + \frac{\alpha}{2}f_{m-1} = 0$. These root is determined from

$$A_{m-1}(\alpha)M\left(\frac{\alpha}{2} + 1, \frac{N}{2}; s_{m-1}\right) = -B_{m-1}(\alpha) \left\{1 + \frac{\alpha - N}{2}\right\} U\left(\frac{\alpha}{2} + 1, \frac{N}{2}; s_{m-1}\right).$$

We remark that through the implicit function theorem, the dependence of A_{m-1} with respect to similarity parameter α is obtained. This dependence is obtained by recurrence, using the results in each previous interval. That is, A_{m-1} depends of A_k ; B_k ; s_k with $k = 1, 2, \dots, m-2$. Similar for B_{m-1} and s_{m-1} .

The asymptotic behaviors of $f(\eta; \alpha)$ are obtained using the asymptotic representations of confluent hypergeometric functions $M(a, b; z)$ and $U(a, b; z)$. These ideas are developed in the next section.

4 Main Results

Since our profiles $f(\eta; \alpha)$ can be defined from (18), their asymptotic behaviors are described by the asymptotic behavior of the Kummer and Tricomi functions, see (38).

We begin considering the asymptotic representation of the Kummer function given by

$$M(a, b; z) = \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a} \left[\sum_{k=0}^N \frac{(a)_k (a-b+1)_k}{k! (-z)^k} + O(z^{-N}) \right] \quad (z < 0). \quad (40)$$

This asymptotic result is valid when $b-a \neq -l$ with $l = 0, 1, 2, \dots$, while in cases $b-a = -l$ the behavior is obtained from (14).

Throughout our proofs, we consider the following confluent hypergeometric functions (see [19] 10.09)

$$V(a, b; z) = \frac{\Gamma(a)}{\Gamma(b)} e^{(b-a)i\pi} M(a, b; z) - \frac{\Gamma(a)}{\Gamma(b-a)} e^{bi\pi} U(a, b; z). \quad (41)$$

The asymptotic representations of our profiles $f(\eta; \alpha)$ are also written in terms of $V(a, b; z)$. We note that the asymptotic representation for $V(a, b; z)$ is given by (see 10.02 in [19])

$$V(a, b; z) = e^z (-z)^{a-b} \left[\sum_{k=0}^N \frac{(b-a)_k (1-a)_k}{k! z^k} + O(z^{-N}) \right] \quad (z < 0). \quad (42)$$

Therefore, using (40) and (42), the asymptotic representation for $U(a, b; z)$ can be obtained from (41).

On the other hand, the function in (41) is also introduced as follows (see 10.03 in [19]):

$$V(a, b; z) = e^z U(b-a, b; z) \quad (43)$$

From this definition, we note that the gaussian-type decays of $f(\eta; \alpha)$ are related with $V(a, b; z)$.

Proof Theorem 1.1: To illustrate our method, we begin studying the existence of $\alpha = \alpha_0$, i.e. η_1 is the unique solution for the equation $\eta f' + \alpha f = 0$. The general case is studied in a similar form. In this proof we assume that $\alpha \neq N$, the other case is analyzed in a similar way, but considering other two linearly independent solution for the Kummer equation.

When η_1 is the unique solution of $\eta f' + \alpha f = 0$ and knowing that $f(\eta; \alpha) = f_2(\eta; \alpha)$ for $\eta > \eta_1$, follows that the asymptotic behavior of $f(\eta; \alpha)$ is given by the asymptotic representation of f_2 .

Assuming $\Gamma(N/2)A_2(\alpha) + e^{-i\pi\alpha/2}\Gamma(\{N - \alpha\}/2)B_2(\alpha) \neq 0$, from the asymptotic behaviors given in (40) and (42) when η is large we get the following asymptotic representation:

$$f_2(\eta; \alpha) \sim \left\{ \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2} - \frac{\alpha}{2})} A_2(\alpha) + e^{-i\frac{\pi}{2}\alpha} B_2(\alpha) \right\} \left(\frac{1 + \gamma}{4} \eta^2 \right)^{-\alpha/2}. \quad (44)$$

If η_1 is the unique root, then we continue analyzing the asymptotic representation for $f_2(\eta; \alpha)$ (the general case is studied below).

Using (31) and (32), the asymptotic representation (44) is written as follows:

$$\begin{aligned} f_2(\eta; \alpha) \sim & -\frac{f_1(\eta_1; \alpha)}{s_1 \mathcal{W}_1} \frac{\alpha}{2} \left\{ \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2} - \frac{\alpha}{2})} \left(1 + \frac{\alpha}{2} - \frac{N}{2} \right) U\left(\frac{\alpha}{2} + 1, \frac{N}{2}; s_1\right) \right. \\ & \left. + e^{-i\frac{\pi}{2}\alpha} M\left(\frac{\alpha}{2} + 1, \frac{N}{2}; s_1\right) \right\} \left((1 + \gamma) \frac{\eta^2}{4} \right)^{-\frac{\alpha}{2}}. \end{aligned}$$

Now, from the definition in (29) follows

$$\begin{aligned} f_2(\eta; \alpha) \sim & -e^{-s_1} (s_1)^{\frac{N}{2}-1} f_1(\eta_1; \alpha) \frac{\alpha}{2} \left\{ \frac{\Gamma(\frac{\alpha}{2})(1 + \frac{\alpha}{2} - \frac{N}{2})}{\Gamma(\frac{N}{2} - \frac{\alpha}{2})} U\left(\frac{\alpha}{2} + 1, \frac{N}{2}; s_1\right) \right. \\ & \left. - \frac{e^{-i\frac{\pi}{2}\alpha} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N}{2})} M\left(\frac{\alpha}{2} + 1, \frac{N}{2}; s_1\right) \right\} \left((1 + \gamma) \frac{\eta^2}{4} \right)^{-\frac{\alpha}{2}}, \end{aligned}$$

with $s_1 = -(1 + \gamma)\eta^2/4$.

Knowing $\frac{\alpha}{2}\Gamma(\frac{\alpha}{2}) = \Gamma(1 + \frac{\alpha}{2})$ and $\frac{1 + \frac{\alpha}{2} - \frac{N}{2}}{\Gamma(\{N - \alpha\}/2)} = -\frac{1}{\Gamma(\frac{N}{2} - \frac{\alpha}{2} - 1)}$, we get

$$\begin{aligned} f_2(\eta; \alpha) \sim & e^{\pi i (s_1)^{\frac{N}{2}-1}} f_1(\eta_1; \alpha) e^{-i\frac{N}{2}\pi} \left\{ \frac{\Gamma(1 + \frac{\alpha}{2}) e^{i\frac{N}{2}\pi}}{\Gamma(\frac{N}{2} - \frac{\alpha}{2} - 1)} U\left(\frac{\alpha}{2} + 1, \frac{N}{2}; s_1\right) + \right. \\ & \left. \frac{e^{i\{\frac{N}{2} - \frac{\alpha}{2} - 1\}\pi} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N}{2})} M\left(\frac{\alpha}{2} + 1, \frac{N}{2}; s_1\right) \right\} e^{-s_1} \left((1 + \gamma) \frac{\eta^2}{4} \right)^{-\frac{\alpha}{2}}. \end{aligned}$$

Finally, from (41) and (43) we obtain

$$f_2(\eta; \alpha) \sim C_2(\alpha) \eta^{-\alpha}, \quad \text{as } \eta \rightarrow \infty, \quad (45)$$

where

$$C_2(\alpha) = ((1 + \gamma)/4)^{\frac{N}{2} - \frac{\alpha}{2} - 1} \eta_1^{N-2} f_1(\eta_1; \alpha) U\left(\frac{N - \alpha}{2} - 1, \frac{N}{2}; \frac{(1 + \gamma)}{4} \eta_1^2\right).$$

Now, we study the existence of similarity exponent α_0 through the sign change of $f_2(\eta; \alpha)$. From Lemma 2.2, for each $\alpha > 0$ we have $f(\eta_1; \alpha) > 0$ and therefore $f_1(\eta_1; \alpha) > 0$. Thus, the change in the asymptotic behavior (45) is given by the condition

$$U\left(\frac{N - \alpha}{2} - 1, \frac{N}{2}; \frac{(1 + \gamma)}{4} \eta_1^2\right) = 0 \quad (46)$$

On the other hand, we know that while η_1 be the unique root of $\eta f' + \alpha f = 0$, we have $f(\eta; \alpha) = f_2(\eta; \alpha)$ for $\eta > \eta_1$. Thus, the change of asymptotic behavior of $f(\eta; \alpha)$ is related with (46). Moreover, the existence of η_2 , the second root of $\eta f' + \alpha f = 0$, is also related with (46), see Lemma 2.4.

Let α^* be a root of (46) with $\eta_1 = \eta_1(\alpha)$ from (24). From (44), we have $\frac{\Gamma(N/2)}{\Gamma(\{N - \alpha\}/2)} A_2(\alpha^*) + e^{-i\pi\alpha^*/2} B_2(\alpha^*) = 0$ and therefore

$$A_2(\alpha^*) = -e^{-i\pi\alpha^*/2} \frac{\Gamma(\{N - \alpha\}/2)}{\Gamma(N/2)} B_2(\alpha^*),$$

Likewise the deduction of (45), we get

$$f_2(\eta; \alpha^*) = -B_2(\alpha^*) \frac{\Gamma(\{N - \alpha^*\}/2)}{\Gamma(\alpha^*/2)} e^{-i\frac{N}{2}\pi} V\left(\frac{\alpha}{2}, \frac{N}{2}; -\frac{1 + \gamma}{4} \eta^2\right)$$

and, knowing the asymptotic representation (42), we obtain

$$f_2(\eta; \alpha^*) = D_2(\alpha^*) \eta^{\alpha^* - N} e^{-\frac{1 + \gamma}{4} \eta^2} [1 + O(\eta^{-2})], \quad \eta \rightarrow \infty,$$

where

$$D_2(\alpha^*) = -(\{1 + \gamma\}/4)^{\frac{\alpha^* - N}{2}} \frac{\Gamma(\{N - \alpha^*\}/2)}{\Gamma(\alpha^*/2)} e^{-i\frac{N}{2}\pi} B_2(\alpha^*)$$

Finally, since α^* is a root of (46) and knowing the representation of $B_2(\alpha)$ in (32), we obtain $f_2(\eta; \alpha^*) > 0$ considering the smallest root such that

$$M\left(\frac{N - \alpha^*}{2} - 1, \frac{N}{2}; \frac{(1 + \gamma)}{4} \eta_1^2\right) \cdot \Gamma(\{N - \alpha^*\}/2) > 0 \quad (47)$$

Under the condition (47), for each $0 \leq \alpha \leq \alpha^*$ we have $f(\eta; \alpha) = f_2(\eta; \alpha)$ when $\eta \geq \eta_1$. Taking $\alpha_0 = \alpha^*$, our first critical exponent $\alpha = \alpha_0$ is obtained. Hence, if $\alpha < \alpha_0$ the asymptotic representation is given in (45), while the asymptotic behavior at $\alpha = \alpha_0$ is (4). For $\alpha = N$, from (14), $M(\frac{\alpha}{2}, \frac{N}{2}; -\frac{(1 + \gamma)}{4} \eta^2)$ has gaussian-type behavior and therefore the asymptotic representation at $f_2(\eta; \alpha)$ is given in (44) considering $A_2(\alpha) = 0$. The comment above is necessary for the description of $f(\eta; \alpha)$ when $\alpha_0 > N$, i.e. when $\gamma > 0$.

The general case is studied in a similar way. For the existence of α_n we assume that $\eta f' + \alpha f = 0$ has exactly $n + 1$ roots $\eta_1 < \eta_2 < \dots < \eta_{n+1}$. Through

the representation of $f(\eta; \alpha)$ given in (18), we note that the existence of α_n is obtained following a similar argument like case α_0 . Under the assumption:

$$\frac{\Gamma(N/2)}{\Gamma(\{N - \alpha\}/2)} A_{n+2}(\alpha) + e^{-i\pi\alpha/2} B_{n+2}(\alpha) \neq 0,$$

and considering that $N - \alpha \neq -l$ ($l = 0, 1, 2, \dots$), the asymptotic representation of $f_{n+2}(\eta; \alpha)$ is given by

$$f_{n+2}(\eta; \alpha) \sim \left(\frac{1+(-1)^n\gamma}{4} \right)^{-\frac{\alpha}{2}} \left\{ \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2}-\frac{\alpha}{2})} A_{n+2}(\alpha) + e^{-i\frac{\pi}{2}\alpha} B_{n+2}(\alpha) \right\} \eta^{-\alpha}$$

Assuming $N - \alpha = -l$ ($l = 0, 1, 2, \dots$), we consider $A_{n+2}(\alpha) = 0$. From the representations given in (39) and (41), we obtain

$$f_{n+2}(\eta; \alpha) \sim C_{n+2}(\alpha) \left(\frac{1+\gamma}{4} \right)^{\frac{N}{2}-\frac{\alpha}{2}-1} \eta^{-\alpha}, \quad \text{as } \eta \rightarrow \infty$$

where

$$C_{n+2}(\alpha) = \eta_{n+1}^{N-2} f_{n+1}(\eta_{n+1}; \alpha) U\left(\frac{N}{2} - \frac{\alpha}{2} - 1, \frac{N}{2}; \frac{1+(-1)^n\gamma}{4} \eta_{n+1}^2\right).$$

In case

$$\frac{\Gamma(N/2)}{\Gamma(\{N - \alpha\}/2)} A_{m+2}(\alpha) + e^{-i\pi\alpha/2} B_{m+2}(\alpha) = 0$$

following similar step as the deduction of (4), we get

$$f_{n+2}(\eta; \alpha^*) = -B_{n+2}(\alpha^*) \frac{\Gamma(\{N - \alpha^*\}/2)}{\Gamma(\alpha^*/2)} e^{-i\frac{N}{2}\pi} V\left(\frac{\alpha}{2}, \frac{N}{2}; -\frac{1+(-1)^n\gamma}{4} \eta^2\right)$$

and, knowing the asymptotic representation (42), we obtain

$$f_{n+2}(\eta; \alpha^*) = D_{n+2}(\alpha^*) \eta^{\alpha^*-N} e^{-\frac{1+(-1)^n\gamma}{4} \eta^2} [1 + O(\eta^{-2})], \quad \eta \rightarrow \infty$$

where

$$D_{n+2}(\alpha^*) = -(\{1 + \gamma\}/4)^{\frac{\alpha^*-N}{2}} \frac{\Gamma(\{N - \alpha^*\}/2)}{\Gamma(\alpha^*/2)} e^{-i\frac{N}{2}\pi} B_{n+2}(\alpha^*)$$

Hence, knowing the representations given in (39), and assuming that α_{n-1} exist, the critical exponent α_n is obtained considering the smallest α^* such that

$$\begin{aligned} M\left(\frac{N-\alpha^*}{2} - 1, \frac{N}{2}; \frac{(1-\gamma)}{4} \eta_1^2\right) &= 0 \\ U\left(\frac{N-\alpha^*}{2} - 1, \frac{N}{2}; \frac{(1+(-1)^n\gamma)}{4} \eta_{n+1}^2\right) &= 0 \\ M\left(\frac{N-\alpha^*}{2} - 1, \frac{N}{2}; \frac{(1+(-1)^n\gamma)}{4} \eta_{n+1}^2\right) \cdot \Gamma(\{N - \alpha^*\}/2) \cdot (-1)^n &> 0, \end{aligned} \quad (48)$$

satisfying $\alpha_{n-1} < \alpha^*$. Finally, if α satisfy $\alpha_{n-1} < \alpha < \alpha_n$, the asymptotic representation is given by (4), finishing our proof. \square

Proof Theorem 1.2: Consider α such that $\alpha_k < \alpha \leq \alpha_{k+1}$. Using the analysis above, $\eta f' + \alpha f = 0$ has exactly $k + 1$ roots $\eta_1 < \eta_2 < \dots < \eta_{k+1}$. Thus, the profile $f(\eta; \alpha)$ is described through $k + 2$ linear Cauchy problems, each defined in $[0, \eta_1[; [\eta_1, \eta_2[; \dots; [\eta_k, \eta_{k+1}[; [\eta_{k+1}, \infty[$, respectively. From Lemma 2.3, $f(\eta; \alpha)$ has exactly k zeros in $[0, \eta_{k+1}[$. Finally, using Lemma 2.4 the profile $f(\eta; \alpha)$ does not change sign in $[\eta_{k+1}, \infty[$, and therefore the proof is finished. \square

5 Extension to a nonlinear diffusion equation

In this section we show how the results presented in Theorems 1.1 and 1.2 can be extended on certain parabolic Bellman equations

$$u_t = F(D^2 u) \quad \text{in } \mathbb{R}^N \times]0, \infty[. \quad (49)$$

Here, the diffusion term is defined by a mapping $F : \mathcal{S}_N \rightarrow \mathbb{R}$, where \mathcal{S}_N denotes the space of symmetric matrices of order $N \times N$ and $D^2 u$ denotes the Hessian matrix of $u(x, t)$ with respect to the spacial variable $x \in \mathbb{R}^N$.

In regard of the general problem (49), a complete analysis on asymptotic behaviors for t large of the solutions was presented in [2], where relation between asymptotic behaviors and self-similar solutions of (49) with gaussian-type decays was obtained. The conditions to this asymptotic behavior is given by the assumption that the positive initial conditions $u(x, 0)$ has spatial decay at most of gaussian-type.

Analysis on (49) can begin considering the Pucci extremal operators. These extremal operators can be introduced as follows (see [9]):

$$\mathcal{M}_{\lambda, \Lambda}^-(A) = \lambda \operatorname{tr}(A^+) - \Lambda \operatorname{tr}(A^-), \quad \mathcal{M}_{\lambda, \Lambda}^+(A) = \Lambda \operatorname{tr}(A^+) - \lambda \operatorname{tr}(A^-), \quad (50)$$

with A^- , A^+ semi-defined positives matrices, such that $A = A^+ - A^-$, $A^- \cdot A^+ = 0$, i.e., the orthogonal decomposition of A .

Thus, when F is uniformly elliptic and positively homogeneous operator, studying the following uniformly parabolic equation

$$u_t = \mathcal{M}_{\lambda, \Lambda}^+(D^2 u) \quad \text{in } \mathbb{R}^N \times]0, \infty[, \quad (51)$$

we obtain super-solutions for (49). Similarly, sub-solutions are obtained considering $\mathcal{M}_{\lambda, \Lambda}^-$. Hence, using comparison results, many qualitative behaviors for (49) can be studied through the solutions of (51).

If we consider that (51) admits solutions of the form (2), the profiles $f(\eta; \alpha)$ satisfy the following nonlinear Cauchy problem:

$$\begin{cases} \sigma(f'')f'' + \sigma(f')\frac{N-1}{\eta}f' &= \frac{\alpha}{2}f' + \frac{\eta}{2}f \\ f(0) &= 1 \\ f'(0) &= 0 \end{cases} \quad (52)$$

Here $\sigma(z)$ is a discontinuous function defined as follows: $\sigma(z) = \Lambda$ if $z > 0$ and $\sigma(z) = \lambda$ if $z < 0$. The analysis of (52) for the case $\alpha = \alpha_0$ was studied in [18]. The problem (52) is analyzed by linear problems, similar to the problem (8). For (52), the ranges of each linear problem are defined using roots $\tilde{\eta}_k$ of $f''(\eta) = 0$ and the roots $\hat{\eta}_k$ of $f'(\eta) = 0$. Being careful with some details, the solutions of (52) can be determined and represented by confluent hypergeometric functions, having powerful tools to obtain results similar as to Theorems 1.1 and 1.2. For $\mathcal{M}_{\lambda, \Lambda}^-$ the result is similar considering in (52) the function σ such that $\sigma(z) = \lambda$ if $z > 0$ and $\sigma(z) = \Lambda$ if $z < 0$.

6 Comments and Conclusions

In this article we developed a method to describe the similarity solution to (1). Using a simple change of variable we obtain an explicit representation of the similarity solution through the confluent hypergeometric functions, obtaining a generalization of the results presented in [5, 16]. We applied the explicit representation to describe the oscillatory and asymptotic behaviors of the similarity solution, obtaining similar results to the convection heat equation (case $\gamma = 0$). The gaussian-type decays show the generation of a new zero for the profiles. This feature can be seen in asymptotic representation (4). Each exponent $\alpha_0 < \alpha_1 < \dots$ in (3) is related with the zeros of Kummer and Tricomi functions. This analysis allows generating new explicit approximations for anomalous exponents, which we hope to develop in a future paper.

Our method can be easy applied on other nonlinear parabolic equations. We believe that through the asymptotic behaviors of similarity solution of (51) it's possible to obtain conditions on u_0 for the existence of global solution to fully nonlinear problems

$$\begin{aligned} u_t &= F(D^2u) + u^p & x \in \mathbb{R}^N, \ t > 0, \ u \geq 0, \ p > 1 \\ u(x, 0) &= u_0(x) & u_0 \geq 0, \ u_0 \neq 0 \end{aligned}$$

These topics are beyond to scope of this article and we leave them for future studies.

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